

Conservative constraint satisfaction re-revisited

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Abstract

Conservative constraint satisfaction problems (CSPs) constitute an important particular case of the general CSP, in which the allowed values of each variable can be restricted in an arbitrary way. Problems of this type are well studied for graph homomorphisms. A dichotomy theorem characterizing conservative CSPs solvable in polynomial time and proving that the remaining ones are NP-complete was proved by Bulatov in 2003. Its proof, however, is quite long and technical. A shorter proof of this result based on the absorbing subuniverses technique was suggested by Barto in 2011. In this paper we give a short elementary prove of the dichotomy theorem for the conservative CSP.

1 Introduction

In a constraint satisfaction problem (CSP) the aim is to find an assignment of values to a given set of variables, subject to specified constraints. The CSP is known to be NP-complete in general. However, certain restrictions on the form of the allowed constraints can lead to problems solvable in polynomial time. Such restrictions are usually imposed by specifying a constraint language, that is, a set of relations that are allowed to be used as constraints. A principal research direction aims to distinguish those constraint languages that give rise to CSPs solvable in polynomial time from those that do not. The dichotomy conjecture [14] suggests that every constraint language gives rise to a CSP that is either solvable in polynomial time or is NP-complete. The dichotomy conjecture is confirmed in a variety

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of particular cases [1, 2, 3, 4, 6, 15, 21], but the general problem remains open.

One of the important versions of the CSP is often referred to as the conservative or list CSP. In a CSP of this type the set of values for each individual variable can be restricted arbitrarily. Restrictions of this type can be studied by considering those constraint languages which contain all possible unary constraints; such languages are also called conservative. Conservative CSPs have been intensively studied for languages consisting of only one binary symmetric relation, that is, graphs; in this case CSP is equivalent to the graph homomorphism problem [11, 12, 13, 15, 19].

In [2, 4] the dichotomy conjecture was confirmed for conservative CSPs. However, the proof given in [2, 4] is quite long and technical, which prompted attempts to find a simpler argument. In [1] Barto gave a simpler proof using the absorbing subuniverses techniques. In the present paper we give another, more elementary, proof that applies the reduction suggested in [20].

As in the majority of dichotomy results the solution algorithm and the proofs heavily use the algebraic approach to the CSP developed in [5, 7, 18, 16]. This approach relates a constraint language to a collection of polymorphisms of the language, that is, operations on the same set that preserves all the relations from the language, and uses polymorphisms of specific types to identify constraint languages solvable in polynomial time. For example, to characterize CSPs on a 2-element set solvable in polynomial time [21] it suffices to consider only 4 types of operations on a 2-element set: constant, semilattice (conjunction and disjunction), majority $((x \wedge y) \vee (y \wedge z) \vee (z \wedge x))$, and affine $(x + y + z)$. The same types of operations characterize the complexity of conservative CSPs, except that constant operations cannot be polymorphisms of conservative languages. In a simplified form the main result we prove is

Theorem 1.1 ([2, 4]) *Let Γ be a constraint language on a set A . The conservative CSP using relations from Γ can be solved in polynomial time if and only if for any 2-element subset $\{a, b\} \subseteq A$ there is an operation f on A , a polymorphism of Γ , such that f on $\{a, b\}$ is either a semilattice operation, or a majority operation, or an affine operation. Otherwise this CSP is NP-complete.*

We give a new nearly complete proof of Theorem 1.1. The only statements we reuse in this paper is Proposition 2.2 that we borrow from [2] and the results of Section 4.2.

2 Definitions and preliminaries

2.1 Constraint satisfaction problems and algebra

By $[n]$ we denote the set $\{1, \dots, n\}$. Let A_1, \dots, A_n be sets, any element of $A_1 \times \dots \times A_n$ is an (n -ary) tuple. Tuples will be denoted in boldface, say, \mathbf{a} , and the i th component of \mathbf{a} will be referred to as $\mathbf{a}[i]$. An n -ary relation over A_1, \dots, A_n is any set of tuples over these sets. For a set $I = \{i_1, \dots, i_k\} \subseteq [n]$, a tuple $\mathbf{a} \in A_1 \times \dots \times A_n$, and a relation $R \subseteq A_1 \times \dots \times A_n$, by $\text{pr}_I \mathbf{a}$ we denote the tuple $(\mathbf{a}[i_1], \dots, \mathbf{a}[i_k])$, the *projection* of \mathbf{a} on I , and $\text{pr}_I R = \{\text{pr}_I \mathbf{b} \mid \mathbf{b} \in R\}$ denotes the projection of R on I . Relation R is said to be a *subdirect product* of A_1, \dots, A_n if $\text{pr}_i R = A_i$ for all $i \in [n]$. Let $I \subseteq [n]$. For $\mathbf{a} \in \text{pr}_I R$ and $\mathbf{b} \in \text{pr}_{[n]-I} R$ by (\mathbf{a}, \mathbf{b}) we denote the tuple \mathbf{c} such that $\mathbf{c}[i] = \mathbf{a}[i]$ if $i \in I$ and $\mathbf{c}[i] = \mathbf{b}[i]$ otherwise.

Let \mathfrak{A} be a collection of finite sets (in this paper we assume \mathfrak{A} to be finite as well). A *constraint satisfaction problem* over \mathfrak{A} is a triple (V, δ, \mathcal{C}) , where V is a (finite) set of *variables*, δ is a *domain function*, $\delta : V \rightarrow \mathfrak{A}$ assigning a domain of values to every variable, and \mathcal{C} is a set of *constraints*. Every constraint is a pair $\langle \mathbf{s}, R \rangle$, where $\mathbf{s} = (v_1, \dots, v_k)$ is a sequence of variables from V (possibly with repetitions) called the *constraint scope*, and R is a relation over $\delta(v_1) \times \dots \times \delta(v_k)$ called the *constraint relation*. A mapping $\varphi : V \rightarrow \bigcup \mathfrak{A}$ that maps every variable v to its domain $\delta(v)$ is called a *solution* if for every $\langle \mathbf{s}, R \rangle \in \mathcal{C}$ we have $\varphi(\mathbf{s}) \in R$.

Let $W \subseteq V$. A *partial solution* of \mathcal{P} on W is a mapping $\varphi : W \rightarrow \bigcup \mathfrak{A}$ such that for every constraint $\langle \mathbf{s}, R \rangle \in \mathcal{C}$, $\mathbf{s} = (v_1, \dots, v_k)$, we have $\varphi(\mathbf{s}') \in \text{pr}_I R$, where $I = \{i_1, \dots, i_\ell\}$ is the set of indices i_s from $[k]$ such that $v_{i_s} \in W$, and $\mathbf{s}' = (v_{i_1}, \dots, v_{i_\ell})$. The set of all partial solutions on set W is denoted by S_W . Problem \mathcal{P} is said to be *3-minimal* if it contains a constraint $\langle W, S_W \rangle$ for every 3-element $W \subseteq V$, and for any $W_1, W_2 \subseteq V$ such that $|W_1| = |W_2| = 3$ and $|W_1 \cap W_2| = 2$, $\text{pr}_{W_1 \cap W_2} S_W = \text{pr}_{W_1 \cap W_2} S_{W_1} \cap \text{pr}_{W_1 \cap W_2} S_{W_2}$. There are standard polynomial time *propagation* algorithms (see, e.g. [10]) to convert any CSP to an equivalent, that is, having the same solutions, 3-minimal CSP.

An introduction into universal algebra and the algebraic approach to CSP can be found in [8, 5, 7, 2]. Here we only mention several key points. For an algebra \mathbb{A} its universe will be denoted by A . Let \mathfrak{A} be a finite collection of finite similar algebras. For a basic or term operation f of the class \mathfrak{A} by $f^{\mathbb{A}}$, $\mathbb{A} \in \mathfrak{A}$, we denote the interpretation of f in \mathbb{A} . Let $\mathbb{A}_1, \dots, \mathbb{A}_k \in \mathfrak{A}$. A relation $R \subseteq A_1 \times \dots \times A_k$ is a *subalgebra* of the direct product $\mathbb{A}_1 \times \dots \times \mathbb{A}_k$, denoted $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_k$, if for any basic operation

f (say, it is n -ary) of \mathfrak{A} and any $\mathbf{a}_1, \dots, \mathbf{a}_n \in R$ the tuple $f(\mathbf{a}_1, \dots, \mathbf{a}_n) = (f^{\mathbb{A}_1}(\mathbf{a}_1[1], \dots, \mathbf{a}_n[1]), \dots, f^{\mathbb{A}_k}(\mathbf{a}_1[k], \dots, \mathbf{a}_n[k]))$ belongs to R . In this case f is also said to be a *polymorphism* of R .

By $\text{CSP}(\mathfrak{A})$ we denote the class of CSP problems $\mathcal{P} = (V, \delta, \mathcal{C})$ such that $\delta(v)$ is the universe of one of the members of \mathfrak{A} , and every constraint relation is a subalgebra of the direct product of the domain algebras. In this paper we assume that the algebras from \mathfrak{A} satisfy certain requirements. An algebra is said to be *conservative* if every subset of its universe is a subalgebra. We only consider classes of conservative algebras. Also, the class \mathfrak{A} will be assumed to be closed under subalgebras. That is, if $\mathbb{A} \in \mathfrak{A}$ then every subalgebra of \mathbb{A} also belongs to \mathfrak{A} . By [5, 7], for any finite \mathfrak{A} the problem $\text{CSP}(\mathfrak{A})$ has the same complexity as \mathfrak{A}' , where \mathfrak{A}' is obtained from \mathfrak{A} by adding all the subalgebras of algebras from \mathfrak{A} . A *unary polynomial* of an algebra \mathbb{A} is a mapping $p : \mathbb{A} \rightarrow \mathbb{A}$, for which there exists a term operation $t(x, y_1, \dots, y_k)$ and elements $a_1, \dots, a_k \in \mathbb{A}$ such that $p(x) = t(x, a_1, \dots, a_k)$. Unary polynomial $p(x)$ is idempotent if $p(p(x)) = p(x)$. The *retract* of \mathbb{A} via polynomial $p(x)$ is the algebra $p(\mathbb{A})$ with the universe $p(A)$, where A is the universe of \mathbb{A} and term operations $p(t)$, where $t(x_1, \dots, x_k)$ is a term operation of \mathbb{A} and $p(t)(a_1, \dots, a_k) = p(t(a_1, \dots, a_k))$ for any $a_1, \dots, a_k \in p(A)$. We will additionally assume that class \mathfrak{A} is closed under retracts. This however does not impose any additional restrictions in the case of conservative algebras, since, as is easily seen, every retract of a conservative algebra is a subalgebra.

A subalgebra \mathbb{A} of a direct product of algebras $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$ is said to be a *subdirect product* if the universe of \mathbb{A} viewed as a relation is a subdirect product of the universes of $\mathbb{A}_1, \dots, \mathbb{A}_n$. For a congruence α of algebra \mathbb{A} and element a by \mathbb{A}/α we denote the factor-algebra of \mathbb{A} and by a^α the block of α containing a .

2.2 Graphs, paths, and the three basic operations

If \mathfrak{A} is a class of conservative algebras closed under subalgebras, then every subalgebra \mathbb{B} of any $\mathbb{A} \in \mathfrak{A}$ belongs to \mathfrak{A} . Therefore, by [21], if $\text{CSP}(\mathfrak{A})$ is polynomial time solvable then, for any 2-element subalgebra \mathbb{B} of \mathbb{A} (we assume $\mathbb{B} = \{0, 1\}$), there exists a term operation $f_{\mathbb{B}}$ of \mathfrak{A} such that $f_{\mathbb{B}}^{\mathbb{B}}$ is one of the operations yielding the tractability of the CSP on a 2-element set: $f_{\mathbb{B}}^{\mathbb{B}}$ is either a *semilattice* (that is conjunction or disjunction) operation, or the *majority* operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, or the *affine* operation $x - y + z \pmod{2}$. Note that the constant operations are not in this list since Γ is conservative. In [4, 2] it was proved that this property is also sufficient for the tractability of $\text{CSP}(\mathfrak{A})$.

Theorem 2.1 ([4, 2]) *Let \mathfrak{A} be a finite class of conservative algebras. The problem $\text{CSP}(\mathfrak{A})$ can be solved in polynomial time if and only if for any $\mathbb{A} \in \mathfrak{A}$ and any 2-element subalgebra \mathbb{B} of \mathbb{A} there is a term operation $f_{\mathbb{B}}$ of \mathfrak{A} such that $f_{\mathbb{B}}^{\mathbb{B}}$ is either semilattice, or majority, or affine. Otherwise $\text{CSP}(\mathfrak{A})$ is NP-complete.*

Let \mathfrak{A} be a finite class of conservative algebras closed under subalgebras that satisfies the conditions of Theorem 2.1. For every $\mathbb{A} \in \mathfrak{A}$, we consider the graph $\mathcal{G}_{\mathfrak{A}}(\mathbb{A})$, an edge-labeled digraph with vertex set A . An edge (a, b) exists and is labeled *semilattice* if there is a term operation $f_{a,b}$ of \mathfrak{A} such that $f_{a,b}^{\mathbb{A}}|_{\{a,b\}}$ is a semilattice operation with $f_{a,b}^{\mathbb{A}}(a, b) = f_{a,b}^{\mathbb{A}}(b, a) = f_{a,b}^{\mathbb{A}}(b, b) = b$, $f_{a,b}^{\mathbb{A}}(a, a) = a$. Edges $(a, b), (b, a)$ exist and are labeled *majority* if neither (a, b) nor (b, a) is semilattice and there is a term operation $f_{a,b}$ such that $f_{a,b}^{\mathbb{A}}|_{\{a,b\}}$ is a majority operation. Edges $(a, b), (b, a)$ exist and are labeled *affine* if none of them is semilattice or majority, and there is a term operation $f_{a,b}$ such that $f_{a,b}^{\mathbb{A}}|_{\{a,b\}}$ is an affine operation. Thus, for each pair $a, b \in A$, either (a, b) or (b, a) is an edge of $\mathcal{G}_{\mathfrak{A}}(\mathbb{A})$; if (a, b) is a majority or affine edge then (b, a) is also an edge with the same label; while if (a, b) is semilattice then the edge (b, a) may not exist. Since \mathfrak{A} is usually fixed, we shall use $\mathcal{G}(\mathbb{A})$ rather than $\mathcal{G}_{\mathfrak{A}}(\mathbb{A})$. The operations of the form $f_{a,b}$ can be considerably unified.

Proposition 2.2 *There are term operations $f(x, y), g(x, y, z), h(x, y, z)$ of \mathfrak{A} such that, for every $\mathbb{A} \in \mathfrak{A}$ and every two-element subset $B \subseteq A$,*

- $f^{\mathbb{A}}|_B$ is a semilattice operation whenever B is semilattice, and $f^{\mathbb{A}}|_B(x, y) = x$ otherwise;
- $g^{\mathbb{A}}|_B$ is a majority operation if B is majority, $g^{\mathbb{A}}|_B(x, y, z) = x$ if B is affine, and $g^{\mathbb{A}}|_B(x, y, z) = f^{\mathbb{A}}|_B(f^{\mathbb{A}}|_B(x, y), z)$ if B is semilattice;
- $h^{\mathbb{A}}|_B$ is an affine operation if B is affine, $h^{\mathbb{A}}|_B(x, y, z) = x$ if B is majority, and $h^{\mathbb{A}}|_B(x, y, z) = f^{\mathbb{A}}|_B(f^{\mathbb{A}}|_B(x, y), z)$ if B is semilattice.

There is also a term operation $p(x, y)$ such that $p^{\mathbb{A}}|_B = f^{\mathbb{A}}|_B$ if B is semilattice, $p^{\mathbb{A}}|_B(x, y) = y$ if B is majority, and $p^{\mathbb{A}}|_B(x, y) = x$ if B is affine.

Using Proposition 2.2 we may assume that all algebras in \mathfrak{A} have only three basic operations. We will normally use \cdot instead of f . Operation \cdot acts non-symmetrically on semilattice edges. This means that every such edge ab is oriented: ab is oriented from a to b if $a \cdot b = b \cdot a = b$; in this case we also write $a \leq b$. Therefore $\mathcal{G}(\mathbb{A})$ is treated as a digraph, in which semilattice edges are oriented, while majority and affine ones are not.

For a relation $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ a digraph $\mathcal{G}(R)$ can be defined in a natural way: tuples $\mathbf{a}, \mathbf{b} \in R$ form a semilattice edge directed from \mathbf{a} to \mathbf{b} if $\mathbf{a}[i] = \mathbf{b}[i]$ or $\mathbf{a}[i]\mathbf{b}[i]$ is a semilattice edge directed from $\mathbf{a}[i]$ to $\mathbf{b}[i]$ for every $i \in [n]$; tuples \mathbf{a}, \mathbf{b} form a majority edge if $\mathbf{a}[i] = \mathbf{b}[i]$ or $\mathbf{a}[i]\mathbf{b}[i]$ is majority for each $i \in [n]$; and \mathbf{a}, \mathbf{b} form an affine edge, if $\mathbf{a}[i] = \mathbf{b}[i]$ or $\mathbf{a}[i]\mathbf{b}[i]$ is affine for every $i \in [n]$. As is easily seen, graph $\mathcal{G}(R)$ is usually not complete, but as we shall see it inherits many properties of the graph $\mathcal{G}(\mathbb{A})$ of a conservative algebra \mathbb{A} .

A sequence of vertices $\mathbf{a}_1, \dots, \mathbf{a}_k$ of $\mathcal{G}(R)$ is a *path* if every $\mathbf{a}_i\mathbf{a}_{i+1}$ is either a semilattice or affine edge.

Lemma 2.3 *Let $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$, $I \subseteq [n]$, and let $\mathbf{a}_1, \dots, \mathbf{a}_k$ be a path in $\mathcal{G}(\text{pr}_I R)$. There are $\mathbf{b}_1, \dots, \mathbf{b}_k \in \text{pr}_{[n]-I} R$ such that $(\mathbf{a}_1, \mathbf{b}_1), \dots, (\mathbf{a}_k, \mathbf{b}_k)$ is a path in R .*

Proof: Observe that for any $\mathbb{A} \in \mathfrak{A}$ and any $a, b \in \mathbb{A}$, the edge $a p(b, a)$ is either semilattice or affine. Therefore, for any $\mathbf{a}, \mathbf{b} \in R$, the pair $\mathbf{a}\mathbf{c}$, where $\mathbf{c} = \mathbf{a} \cdot p(\mathbf{b}, \mathbf{a})$, is semilattice, while $\mathbf{a}\mathbf{d}$, where $\mathbf{d} = p(\mathbf{b}, \mathbf{a}) \cdot \mathbf{a}$, is affine.

Take any $\mathbf{c}_1, \dots, \mathbf{c}_k \in \text{pr}_{[n]-I} R$ such that $(\mathbf{a}_i, \mathbf{c}_i) \in R$ and define $\mathbf{b}_1, \dots, \mathbf{b}_k$ as follows: $\mathbf{b}_1 = \mathbf{c}_1$, if $\mathbf{a}_i\mathbf{a}_{i+1}$ is semilattice then $\mathbf{b}_{i+1} = \mathbf{b}_i \cdot p(\mathbf{c}_{i+1}, \mathbf{b}_i)$, and if $\mathbf{a}_i\mathbf{a}_{i+1}$ is affine then $\mathbf{b}_{i+1} = p(\mathbf{c}_{i+1}, \mathbf{b}_i) \cdot \mathbf{b}_i$. As is easily seen, $\mathbf{b}_1, \dots, \mathbf{b}_k$ satisfy the conditions of the lemma. \square

A set $S \subseteq R$ is said to be *connected* if there is a path from every element in S to every other element in S .

3 Properties of labeled graph of algebras

3.1 As-components, linked relations, and connectivity

Let $\mathbb{A} \in \mathfrak{A}$ be a conservative algebra. A set $B \subseteq A$ is called an *as-component* (for affine-semilattice) if for any $a \in A$ and $b \in A - B$ the edge ba is either majority or semilattice directed from b to a , see Fig. 3.1. Since as-components are defined in terms of the graph $\mathcal{G}(\mathbb{A})$, this definition can be naturally generalized to as-components of relations.

Let $R \leq \mathbb{A} \times \mathbb{B}$, where \mathbb{A}, \mathbb{B} are subdirect products of conservative algebras. By $\text{tol}_1(R)$ we denote the congruence of \mathbb{A} defined as the transitive close of the set $\{(a, b) \in \mathbb{A}^2 \mid \text{there is } c \in \mathbb{B} \text{ with } (a, c), (b, c) \in R\}$. Then $\text{tol}_2(R)$ denotes the congruence on \mathbb{B} defined in a similar way. Relation R is said to be *linked* if $\text{tol}_1(R), \text{tol}_2(R)$ are total relations.

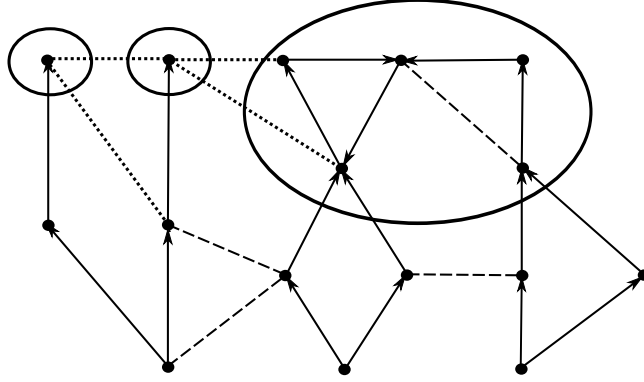


Figure 3.1: As-components. Solid lines represent semilattice edges, dashed lines represent affine edges, dotted lines represent majority edges; the edges that are not shown are majority; as-components are encircled.

Lemma 3.1 *Let $R \leq \mathbb{A} \times \mathbb{B}$, and A', B' be as-components of \mathbb{A}, \mathbb{B} , respectively, such that $R' = R \cap (A' \times B') \neq \emptyset$. Then R' is a subdirect product of A', B' .*

Proof: Let $A'' = \text{pr}_1 R' \subseteq A'$. If $A'' \neq A'$, there are $a \in A''$ and $a' \in A' - A''$ such that $a \leq a'$ or aa' is an affine edge. Take $(a, b), (a', b') \in R$ with $b \in B'$. As is easily seen, $p\left(\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} a' \\ b \end{pmatrix} \in R$, since $b' \notin B'$, implying $a' \in A''$. \square

Lemma 3.2 *Let $R \leq \mathbb{A} \times \mathbb{B}$, and let A', B' be as-components of \mathbb{A}, \mathbb{B} , respectively, such that there is $a \in A'$ with $\{a\} \times B' \subseteq R$. Then $A' \times B' \subseteq R$.*

Proof: By Lemma 3.1 $R \cap (A' \times B')$ is a subdirect product of A', B' . Therefore, if $A' \times B' \not\subseteq R$ there are $b, c \in A', d, e \in B'$ such that $(b, d), (b, e), (c, e) \in R, (c, d) \notin R, b \leq c$ or bc is affine, and $e \leq d$ or ed is affine. If at least one of these two edges is not affine, we have $\begin{pmatrix} c \\ d \end{pmatrix} \in \left\{ p\left(\begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} c \\ e \end{pmatrix}\right), p\left(\begin{pmatrix} c \\ e \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right) \right\}$. If both edges are affine then $\begin{pmatrix} c \\ d \end{pmatrix} = h\left(\begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} c \\ e \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right)$, a contradiction. \square

Lemma 3.3 *Let \mathbb{A}, \mathbb{B} be subdirect products of conservative algebras and let $R \leq \mathbb{A} \times \mathbb{B}$ be subdirect and linked. Let also A', B' be as-components of \mathbb{A}, \mathbb{B} , respectively, such that $R \cap (A' \times B') \neq \emptyset$. Then $A' \times B' \subseteq R$.*

Proof: We prove by induction on the size of \mathbb{A}, \mathbb{B} . The base case of induction, when $|\mathbb{A}| = 1$ or $|\mathbb{B}| = 1$, is obvious.

Take any $\mathbf{b} \in \mathbb{A}$ and construct a sequence of subalgebras B_1, \dots, B_k such that $B_i \subseteq \mathbb{A}$ if i is odd and $B_i \subseteq \mathbb{B}$ if i is even, as follows: $B_1 = \{\mathbf{b}\}$, $B_i = R[B_{i-1}] = \{\mathbf{d} \mid (\mathbf{c}, \mathbf{d}) \in R \text{ for some } \mathbf{c} \in B_{i-1}\}$ if i is odd, and $B_i = R^{-1}[B_{i-1}] = \{\mathbf{c} \mid (\mathbf{c}, \mathbf{d}) \in R \text{ for some } \mathbf{d} \in B_{i-1}\}$ otherwise. By construction for each $i \leq k$ the relation $R_i = R' \cap (B_i \times B_{i+1})$ (or $R_i = R' \cap (B_{i+1} \times B_i)$) is linked. Let k be the maximal with $B_k \subset \mathbb{A}$ or $B_k \subset \mathbb{B}$. Without loss of generality we assume $B_k \subset \mathbb{A}$. Set $\mathbb{A}'' = B_k$. Thus there exists $\mathbb{A}'' \subset \mathbb{A}$ such that $R' = R \cap (\mathbb{A}'' \times \mathbb{B}) \subseteq \mathbb{A}'' \times \mathbb{B}$ is linked and subdirect. Choose a minimal subalgebra \mathbb{A}'' with this property. We show that there is $\mathbf{a} \in \mathbb{A}''$ such that $\{\mathbf{a}\} \times B' \subseteq R$.

If there is an as-component C of \mathbb{A}'' such that $R \cap (C \times B') \neq \emptyset$ then $C \times B' \subseteq R$ by induction hypothesis, and the claim follows. Let $D = R^{-1}[B']$. If D contains no elements from an as-component, there are $\mathbf{b} \in D$ and $\mathbf{c} \in \mathbb{A} - D$ such that $\mathbf{b}\mathbf{c}$ is a semilattice or affine edge. Take $\mathbf{b}' \in B'$ and $\mathbf{c}' \in \mathbb{B}$ such that $(\mathbf{b}, \mathbf{b}'), (\mathbf{c}, \mathbf{c}') \in R$. Let

$$\begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = p \left(\begin{pmatrix} \mathbf{c} \\ \mathbf{c}' \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ \mathbf{b}' \end{pmatrix} \right), \quad \text{and} \quad \begin{pmatrix} \mathbf{c}'' \\ \mathbf{d}' \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{b}' \end{pmatrix} \cdot \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}, \quad \mathbf{c}'' \in \{\mathbf{b}, \mathbf{c}\}.$$

Suppose $\mathbb{B} \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_k$. Then for any $i \in [k]$ the pair $\mathbf{b}'[i]\mathbf{d}[i]$ is a semilattice or affine edge. If there is no semilattice edge of this form then $\mathbf{b}'\mathbf{d}$ is an affine edge, implying $\mathbf{d} \in B'$, and $\mathbf{c} \in D$, a contradiction. Otherwise $\mathbf{b}'\mathbf{d}'$ is a semilattice edge and $\mathbf{d}'\mathbf{d}$ is an affine one, hence $\mathbf{d} \in B'$, a contradiction again.

Let now $\mathbf{a} \in \mathbb{A}''$ be such that $\{\mathbf{a}\} \times B' \subseteq R$. If $\mathbf{a}' \in A'$, we are done. Otherwise take any $\mathbf{b}' \in A'$ with $R[\mathbf{b}] \cap B' \neq \emptyset$, and set $\mathbf{b} = p(\mathbf{a}, \mathbf{b}')$. As before, it is easy to see that $\mathbf{b} \in A'$. Moreover, $p(\mathbf{a}, \mathbf{b}) = \mathbf{b}$. Let also $B'' = B' \cap R[\mathbf{b}]$. If $B'' \neq B'$, there is $\mathbf{c} \in B' - B''$ and $\mathbf{d} \in B''$ such that $\mathbf{d}\mathbf{c}$ is a semilattice or affine edge. Then

$$p \left(\begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix} \right) = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix},$$

a contradiction. Thus \mathbf{a} can be chosen from A' . The proof is now completed by Lemma 3.2. \square

Lemma 3.4 *Let $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ for conservative algebras $\mathbb{A}_1, \dots, \mathbb{A}_n$, and let A'_i be an as-component of \mathbb{A}_i for $i \in [n]$, such that $(a_1, \dots, a_n) \in R$ for some $a_i \in A'_i$, $i \in [n]$. Then $R' = R \cap (A'_1 \times \dots \times A'_n)$ is a subdirect product of the A'_i and R' is an as-component of R .*

Proof: Let us first suppose that $\mathbb{A}_1, \dots, \mathbb{A}_n$ are simple. We prove the result by induction on n . The trivial case $n = 1$ gives the base case of induction. Otherwise, we consider R as a binary relation, a subdirect product of $\mathbb{A} = \text{pr}_{[n-1]}R$ and \mathbb{A}_n .

Let $\mathbf{a}, \mathbf{b} \in R'$, $\mathbf{a}' = \text{pr}_{[n-1]}\mathbf{a}$, $\mathbf{b}' = \text{pr}_{[n-1]}\mathbf{b}$, and $a = \mathbf{a}[n]$, $b = \mathbf{b}[n]$. By the induction hypothesis there is a path $\mathbf{a}' = \mathbf{a}'_1, \mathbf{a}'_2, \dots, \mathbf{a}'_k = \mathbf{b}'$ in $\text{pr}_{[n-1]}R'$. There are two cases. If R is linked (as a subdirect product of $\mathbb{A} \times \mathbb{A}_n$, then $\text{pr}_{[n-1]}R' \times A'_n \subseteq R'$. Otherwise, as R is not linked and \mathbb{A}_n is simple, for every $\mathbf{c}' \in \mathbb{A}$ there is a unique $c \in \mathbb{A}_n$ such that $(\mathbf{c}', c) \in R$. In particular, there are unique a_1, \dots, a_k such that $(\mathbf{a}_i, a_i) \in R$. It is not hard to see that if $\mathbf{a}_i \mathbf{a}_{i+1}$ is a semilattice (affine) edge, so is $a_i a_{i+1}$, because otherwise \mathbf{a}_i or \mathbf{a}_{i+1} has more than one extension. Thus $(\mathbf{a}_1, a_1), \dots, (\mathbf{a}_k, a_k)$ is a path from \mathbf{a} to \mathbf{b} .

Suppose that not all of the algebras $\mathbb{A}_1, \dots, \mathbb{A}_n$ are simple. We prove the lemma by induction on the number of non-simple factors and their size.

We start with a couple of simple observations. If \mathbb{A} is a conservative algebra and α is its congruence, then \mathbb{A}/α is also a conservative algebra. Moreover, if $\overline{a}\overline{b}$, $\overline{a}, \overline{b} \in \mathbb{A}/\alpha$ is a semilattice (majority, affine) edge of \mathbb{A}/α then for any $a \in \overline{a}, b \in \overline{b}$ the edge ab is also semilattice (respectively, majority, affine). It follows immediately from the observation that if $m \in \{f, g, h\}$ then $\{a, b\}$ is closed under m , and $m(x, y, z) = a$ for $x, y, z \in \{a, b\}$ if and only if $m(x^\alpha, y^\alpha, z^\alpha) = \overline{a}$.

Suppose that \mathbb{A}_n is not simple and α is its maximal congruence. From the observation above it follows that $A''_n = \{a^\alpha \mid a \in A'_n\}$ is an as-component of \mathbb{A}_n/α . Consider the relation $S = \{(a_1, \dots, a_{n-1}, a_n^\alpha) \mid (a_1, \dots, a_n) \in R\}$. By the induction hypothesis $S' = \{(a_1, \dots, a_{n-1}, a_n^\alpha) \mid (a_1, \dots, a_n) \in R'\}$ is connected and is an as-component of S . Take $\mathbf{a}, \mathbf{b} \in R'$ and let \mathbf{a}', \mathbf{b}' be the corresponding tuples from S' . Then there is a path $\mathbf{a}' = \mathbf{a}'_1, \mathbf{a}'_2, \dots, \mathbf{a}'_k = \mathbf{b}'$. For each $i \in [k]$ pick a tuple $\mathbf{a}_i \in R'$ such that $\mathbf{a}_i[n] \in \mathbf{a}'_i[n]$. By the observation above, if $\mathbf{a}'_i \mathbf{a}'_{i+1}$ is a semilattice (affine) edge, so is $\mathbf{a}_i[n] \mathbf{a}_{i+1}[n]$, and $\mathbf{a}_i \mathbf{a}_{i+1}$, as well. The sequence $\mathbf{a}_1, \dots, \mathbf{a}_k$ is a path from \mathbf{a} to \mathbf{b} . \square

3.2 Rectangularity

Let $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ and let $A'_i \subseteq \mathbb{A}_i$, $A'_j \subseteq \mathbb{A}_j$ be as-components of \mathbb{A}_i , \mathbb{A}_j , respectively. Positions i and j are said to be A'_i, A'_j -related if $\mathbf{a}[i] \in A'_i$ if and only if $\mathbf{a}[j] \in A'_j$, for any $\mathbf{a} \in R$. A set $I \subseteq [n]$ is called a *strand* with respect to as-component A'_1, \dots, A'_n of $\mathbb{A}_1, \dots, \mathbb{A}_n$, respectively, if it is maximal such that any $i, j \in I$ are A'_i, A'_j -related. As is easily seen, the strands with respect to A'_1, \dots, A'_n form a partition of $[n]$.

Lemma 3.5 *Let $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ and let A'_1, \dots, A'_n be as-components of $\mathbb{A}_1, \dots, \mathbb{A}_n$, respectively, such that $R \cap (A'_1 \times \dots \times A'_n) \neq \emptyset$. Let also I_1, \dots, I_k be the partition of $[n]$ into strands with respect to A'_1, \dots, A'_n and $R_i = \text{pr}_{I_j} R \cap \prod_{\ell \in I_j} A'_\ell$. Then $R_1 \times \dots \times R_k \subseteq R$.*

Proof: We proceed by induction on n . If there is only one strand with respect to A'_1, \dots, A'_n , say, if $n = 1$, there is nothing to prove. So, suppose that there are at least two strands. There are $i, j \in [n]$ and $\mathbf{a}, \mathbf{a}' \in R$ such that $\mathbf{a} \in A'_1 \times \dots \times A'_n$, $\mathbf{a}'[i] \in A'_i$ and $\mathbf{a}'[j] \in A_j - A'_j$. Let $J \subseteq [n]$ be the set of all $\ell \in [n]$ with $\mathbf{a}'[\ell] \in A_\ell - A'_\ell$. Choose \mathbf{a}' such that J is minimal. Without loss of generality, $J = [s]$ for $s < n$. Set $\mathbf{c} = \text{pr}_{[n]-J} \mathbf{a}$, $\mathbf{c}' = \text{pr}_{[n]-J} \mathbf{a}'$ and $\mathbf{b} = \text{pr}_J \mathbf{a}$, $\mathbf{b}' = \text{pr}_J \mathbf{a}'$.

We show first that these tuples can be chosen such that $\mathbf{c} = \mathbf{c}'$. Let $A' = \text{pr}_{[n]-J} R \cap (A'_{s+1} \times \dots \times A'_n)$ and $B' = \text{pr}_J R \cap (A'_1 \times \dots \times A'_s)$. By Lemma 3.4 A' is an as-component of $\text{pr}_{[n]-J} R$ and B' is an as-component of $\text{pr}_J R$. Since $\mathbf{c}, \mathbf{c}' \in A'$, there is a path $\mathbf{c} = \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_k = \mathbf{c}'$. Choose some $\mathbf{b}_1, \dots, \mathbf{b}_k \in \text{pr}_J R$ such that $\mathbf{b}_1 = \mathbf{b}$, $\mathbf{b}_k = \mathbf{b}'$ and $(\mathbf{b}_i, \mathbf{c}_i) \in R$ for $i \in [k]$. There is i such that $\mathbf{b}_i \in B'$, but $\mathbf{b}_{i+1} \in \text{pr}_J R - B'$. Observe that $\mathbf{b}_i \leq \mathbf{d} = \mathbf{b}_i \mathbf{b}_{i+1}$, and $\mathbf{d} \mathbf{d}'$, $\mathbf{d}' = h(\mathbf{d}, \mathbf{d}, \mathbf{b}_{i+1})$, is an affine edge. Therefore $\mathbf{d}, \mathbf{d}' \in B'$. Then

if $\mathbf{c}_i \mathbf{c}_{i+1}$ is semilattice then $\begin{pmatrix} \mathbf{d} \\ \mathbf{c}_{i+1} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_i \\ \mathbf{c}_i \end{pmatrix} \cdot \begin{pmatrix} \mathbf{b}_{i+1} \\ \mathbf{c}_{i+1} \end{pmatrix}$ belongs to R , or
if $\mathbf{c}_i \mathbf{c}_{i+1}$ is affine then $\begin{pmatrix} \mathbf{d}' \\ \mathbf{c}_{i+1} \end{pmatrix} = h\left(\begin{pmatrix} \mathbf{b}_i \\ \mathbf{c}_i \end{pmatrix} \cdot \begin{pmatrix} \mathbf{b}_{i+1} \\ \mathbf{c}_{i+1} \end{pmatrix}, \begin{pmatrix} \mathbf{b}_i \\ \mathbf{c}_i \end{pmatrix} \cdot \begin{pmatrix} \mathbf{b}_{i+1} \\ \mathbf{c}_{i+1} \end{pmatrix}, \begin{pmatrix} \mathbf{b}_{i+1} \\ \mathbf{c}_{i+1} \end{pmatrix}\right)$
belongs to R .

Either way, $\mathbf{c} = \mathbf{c}'$ can be chosen to be \mathbf{c}_{i+1} , and $\mathbf{b} = \mathbf{b}_i$ and \mathbf{b}' to be \mathbf{d} or \mathbf{d}' .

We consider R as a subdirect product of pr_J and $\text{pr}_{[n]-J} R$. Recall that $\text{tol}_1(R)$ denotes the congruence generated by all pairs $(\mathbf{d}, \mathbf{d}') \in (\text{pr}_J R)^2$ that have a common extension $\mathbf{e} \in \text{pr}_{[n]-J} R$ with $(\mathbf{d}, \mathbf{e}), (\mathbf{d}', \mathbf{e}) \in R$. By what is already proved $\text{tol}_1(R)$ is nontrivial, and there are $\mathbf{b} \in B'$ and $\mathbf{b}' \notin B'$ with $(\mathbf{b}, \mathbf{b}') \in \text{tol}_1(R)$. We prove that B' is in a $\text{tol}_1(R)$ -block. For

elements \mathbf{b}, \mathbf{b}' we take the ones found in the previous paragraph; there is also $\mathbf{c} \in A'$ such that $(\mathbf{b}, \mathbf{c}), (\mathbf{b}', \mathbf{c}) \in R$. Now, if $\alpha = \text{tol}_1(R)$ is nontrivial on B' , choose $\mathbf{d} \in B'$ from a different α -block than \mathbf{b} , and such that $\mathbf{d}^\alpha \mathbf{b}^\alpha$ is either semilattice or affine.

First, note that for any $i \in J$ the edge $\mathbf{b}'[i]\mathbf{d}[i]$ is either semilattice or majority. Indeed, suppose this is not the case. If $\mathbf{d}[i]\mathbf{b}'[i]$ is semilattice or affine then $\mathbf{b}'[i] \in A'_i$, a contradiction with the construction. Therefore, $p(\mathbf{b}', \mathbf{d}) = \mathbf{d}$, while $p(\mathbf{b}'^\alpha, \mathbf{d}^\alpha) = \mathbf{b}^\alpha$, a contradiction again.

To complete the proof it remains to apply the lemma to $\text{pr}_J R$ and $\text{pr}_{[n]-J} R$. \square

4 Solving conservative CSPs

Let \mathfrak{A} be a finite class of conservative algebras closed under subalgebras and retracts. For example, as we noted \mathfrak{A} can be the set of all subalgebras of a finite conservative algebra. In this section we present an algorithm solving $\text{CSP}(\mathfrak{A})$. We start with two reductions of the problem.

4.1 The as-component exclusion reduction

The first reduction converts the problem to a number of CSP instances in which every domain is an as-component, and then either provides a solution, or allows to eliminate some elements from some of the original domains.

Let $\mathcal{P} = (V, \delta, \mathcal{C})$ be a $\text{CSP}(\mathfrak{A})$ instance. Choose as-components $A'_v \subseteq \delta(v)$ for each $v \in V$ such that for any constraint $\langle (v_1, \dots, v_n), R \rangle$ the set $R \cap (A'_{v_1} \times \dots \times A'_{v_n})$ is nonempty. We call such a collection of as-components a *consistent collection*. A strand of \mathcal{P} with respect to $A'_v, v \in V$, is a maximal set $W \subseteq V$ such that for any partition W_1, W_2 of W some $w_1 \in W_1, w_2 \in W_2$ are in the same strand with respect to $A'_{v_1}, \dots, A'_{v_n}$ of a constraint $\langle (v_1, \dots, v_n), R \rangle \in \mathcal{C}$. Let W_1, \dots, W_k be the partition of V into strands with respect to $A'_v, v \in V$. For $i \in [k]$ denote by \mathcal{P}_i the problem instance $(W_i, \delta'_i, \mathcal{C}_i)$, where $\delta'_i : W_i \rightarrow \mathfrak{A}$ with $\delta'_i(v) = A'_v$, and for each $\langle (v_1, \dots, v_n), R \rangle \in \mathcal{C}$ we include into \mathcal{C}_i the constraint $\langle (v_{i_1}, \dots, v_{i_\ell}), \text{pr}_{\{i_1, \dots, i_\ell\}} R \rangle$ and i_j is the positions of $v_j \in W_i$.

Lemma 4.1 *If every \mathcal{P}_i has a solution then \mathcal{P} has a solution.*

Proof: Let φ_i be a solution of \mathcal{P}_i . Then applying Lemma 3.5 to each constraint relation of \mathcal{P} we conclude that φ such that $\varphi(v) = \varphi_i(v)$ whenever

$v \in W_i$ is a solution for \mathcal{P} . □

If for some $i \leq k$ the problem \mathcal{P}_i has no solution, then \mathcal{P} has no solution φ with $\varphi(v) \in A'_v$ for any $v \in W_i$. Therefore, \mathcal{P} can be reduced to a smaller problem $(V, \delta', \mathcal{C}')$, where $\delta'(v) = \delta(v) - A'_v$ if $v \in W_i$ and $\delta'(v) = \delta(v)$ otherwise; and every constraint relation R of \mathcal{P} is obtained from the corresponding constraint relation of \mathcal{P} by restricting it to the new domains.

It remains to show that such a consistent collection of as-components always exists, and to demonstrate how it can be found.

Let $W \subseteq V$. A *partial consistent collection* on W is a collection of as-components $A'_v \subseteq \delta(v)$ for each $v \in W$ such that for any constraint $\langle \mathbf{s}, R \rangle$, where $\mathbf{s} \cap W = (v_1, \dots, v_n)$ the set $\text{pr}_{\mathbf{s} \cap W} R \cap (A'_{v_1} \times \dots \times A'_{v_n})$ is nonempty.

Proposition 4.2 *Let $\mathcal{P} = (V, \delta, \mathcal{C})$ be a 3-minimal instance and $W \subseteq V$. Then any partial consistent collection on W can be extended to a consistent collection.*

Observe that Proposition 4.2 implies that a consistent collection always exists (it suffices to start with empty W). It also gives a method of finding a consistent collection: Let $V = \{v_1, \dots, v_n\}$ and choose any as-component A'_{v_1} . Then, if a partial consistent collection $A'_{v_1}, \dots, A'_{v_k}$ is chosen, Proposition 4.2 guarantees that we can find $A'_{v_{k+1}}$ such that $A'_{v_1}, \dots, A'_{v_k}, A'_{v_{k+1}}$ is partial consistent collection.

We start with a statement that is quite similar to Proposition 4.2, but uses relations rather than CSP instances. (Partial) consistent collections for relations are defined as follows: Let $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$, as-components A'_1, \dots, A'_n is a consistent collection if for any $i, j \in [n]$ the set $\text{pr}_{i,j} R \cap (A'_i \times A'_j)$ is non-empty.

Lemma 4.3 *Let R be an $(n\text{-ary})$ relation and $I \subseteq [n]$. For any $\mathbf{a} \in \text{pr}_I R$ such that $\mathbf{a}[i]$, $i \in I$, belongs to an as-component, there is $\mathbf{b} \in R$ such that $\mathbf{b}[i]$, $i \in [n]$, belongs to an as-component and $\mathbf{b}[i] = \mathbf{a}[i]$ for $i \in I$.*

Proof: Consider R as a subdirect product of $R_1 = \text{pr}_I R$ and $R_2 = \text{pr}_{[n]-I} R$. By Lemma 3.4 \mathbf{a} belongs to an as-component of R_1 , and it suffices to find \mathbf{b} in an as-component of R_2 such that $(\mathbf{a}, \mathbf{b}) \in R$.

Let $(\mathbf{a}, \mathbf{b}) \in R$ for some $\mathbf{b} \in R_2$. If $\mathbf{b}[i]$ belongs to an as-component, we may replace I with $I \cup \{i\}$, so assume $\mathbf{b}[i]$ does not belong to an as-component for $i \in [n] - I$. Take $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2) \in R$ with $\mathbf{c}_1 \in R_1$ and \mathbf{c}_2 from an as-component of R_2 . As $\mathbf{b}[i]$ is not in any as-component, $\mathbf{b}[i]\mathbf{c}_2[i]$ is a

semilattice or majority edge for $i \in [n] - I$. Letting $\mathbf{d} = \mathbf{b} \cdot \mathbf{c}_2$ we have that \mathbf{bd} is a semilattice edge and \mathbf{dc}_2 is a majority edge. Observe that

$$\begin{pmatrix} \mathbf{a} \cdot \mathbf{c}_1 \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} \in R \quad \text{and} \quad \begin{pmatrix} p(\mathbf{a} \cdot \mathbf{c}_1, \mathbf{c}_1) \\ \mathbf{c}_2 \end{pmatrix} = p \left(\begin{pmatrix} \mathbf{a} \cdot \mathbf{c}_1 \\ \mathbf{d} \end{pmatrix}, \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} \right) \in R,$$

and that $p(\mathbf{a} \cdot \mathbf{c}_1, \mathbf{c}_1)$ belongs to the same as-component as \mathbf{a} . Thus, by Lemma 3.4 $(\mathbf{a}, \mathbf{c}_3) \in R$ for some \mathbf{c}_3 from the same as-component as \mathbf{c}_2 . \square

Lemma 4.3 implies that for any relation there is a consistent collection. Indeed, if $\mathbf{a} \in R$ is such that $\mathbf{a}[i]$ belongs to an as-component A'_i , then A'_1, \dots, A'_n is a consistent collection.

Lemma 4.4 *Let A'_1, \dots, A'_n be a consistent collection for an n -ary relation R . Then $(A'_1 \times \dots \times A'_n) \cap R \neq \emptyset$.*

Proof: We prove by induction that for any $I \subseteq [n]$ there is $\mathbf{a} \in R$ such that $\mathbf{a}[i] \in A'_i$ for $i \in I$. Since A'_1, \dots, A'_n is a consistent collection, the statement is true for any I with $|I| \leq 2$. Suppose it is true for any $J \subseteq [n]$ such that $|J| < |I|$. Without loss of generality assume $1, 2, 3 \in I$. Let $J_1 = I - \{1\}$, $J_2 = I - \{2\}$, $J_3 = I - \{3\}$, and let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in R$ such that $\mathbf{a}_j[i] \in A'_i$ for all $i \in J_j$. If one of $\mathbf{a}_j[j] \in A'_j$, $j \in \{1, 2, 3\}$, then we are done; assume this is not the case. By Lemma 3.4 $(A'_1 \times A'_3 \times \dots \times A'_n) \cap \text{pr}_{\{1,3,\dots,n\}} R$ and $(A'_1 \times A'_2 \times A'_4 \times \dots \times A'_n) \cap \text{pr}_{\{1,2,4,\dots,n\}} R$ are subdirect products of A'_1, A'_3, \dots, A'_n and $A'_1, A'_2, A'_4, \dots, A'_n$, respectively. Therefore $\mathbf{a}_2, \mathbf{a}_3$ can be chosen so that $\mathbf{a}_1[3] = \mathbf{a}_2[3]$ and $\mathbf{a}_1[2] = \mathbf{a}_3[2]$. While $\mathbf{a}_1[i], \mathbf{a}_2[i] \in A'_i$ for all $i \in \{3, \dots, n\}$, $\mathbf{a}_2[1] \in A'_1$, and $\mathbf{a}_1[1] \notin A'_1$, by Lemma 3.5

$$(A'_1 \times A'_3 \times \dots \times A'_n) \cap \text{pr}_{\{1,3,\dots,n\}} R = A'_1 \times \left[(A'_3 \times \dots \times A'_n) \cap \text{pr}_{\{3,\dots,n\}} R \right].$$

Hence, \mathbf{a}_2 can be assumed such that $\mathbf{a}_2[1] = \mathbf{a}_3[1]$.

If $\mathbf{a}_j[j]\mathbf{a}_k[j]$ is a semilattice edge for some $j, k \in \{1, 2, 3\}$ then the tuple $\mathbf{a}_j\mathbf{a}_k$ satisfies the required conditions. It remains to consider the case when $\mathbf{a}_j[j]\mathbf{a}_k[j]$ is a majority edge for any $j, k \in \{1, 2, 3\}$. Consider $\mathbf{b} = g(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$. As $\mathbf{a}_1[i], \mathbf{a}_2[i], \mathbf{a}_3[i] \in A'_i$ for $i \in \{4, \dots, n\}$, we have $\mathbf{b}[i] \in A'_i$ in this case. Since $\mathbf{a}_2[1] = \mathbf{a}_3[1]$ and $\mathbf{a}_1[1]\mathbf{a}_2[1]$ is a majority edge, $\mathbf{b}[1] = \mathbf{a}_2[1]$. Similarly, $\mathbf{b}[2] = \mathbf{a}_1[2] \in A'_2$ and $\mathbf{b}[3] = \mathbf{a}_1[3] \in A'_3$. \square

Corollary 4.5 *Let $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$, and let A'_1, \dots, A'_{n-1} be a partial consistent collection, $A'_i \subseteq A_i$. Then it can be extended to a consistent collection for R .*

Proof: By Lemma 4.4 there is $\mathbf{a} \in (A'_1 \times \dots \times A'_{n-1}) \cap \text{pr}_{[n-1]} R$. Therefore, by Lemma 4.3 $(\mathbf{a}, a) \in R$ for some a from an as-component of \mathbb{A}_n . \square

By $S_{v,w}, S_{u,v,w}$ we denote the sets of partial solutions of \mathcal{P} on $\{v, w\}$ and $\{u, v, w\}$, respectively.

Proof: (of Proposition 4.2) The proof we give here is a modification of the proof of Theorem 3.5 from [17].

Suppose $\mathcal{P} = (V, \delta, \mathcal{C})$ is a minimal instance that does not satisfy the conclusion of the proposition. Since we assume \mathcal{P} 3-minimal, $|V| > 3$. Pick $v \in V$; our assumption implies that $\mathcal{P}_{V-\{v\}}$ satisfies the conclusion of the proposition, but there is a consistent collection $\{A'_w \subseteq \delta(w) \mid w \in W = V - \{v\}\}$ such that it cannot be extended to a consistent collection including some $A'_v \subseteq \delta(v)$.

Let $\mathcal{C} = \{\langle \mathbf{s}_1, R_1 \rangle, \dots, \langle \mathbf{s}_q, R_q \rangle\}$. To obtain the desired contradiction we shall construct a problem \mathcal{P}' which also has q constraints, with the same constraint relations, but with different constraint scopes.

We define the set of variables of \mathcal{P}' to be the union of $\{v'\}$ and q disjoint copies W_1, \dots, W_q of W , where $W_i = \{w_1^i, \dots, w_k^i\}$. Now, for each $i \in [q]$, we define a mapping $f_i: W \rightarrow W_i$ by setting $f_i(w_j) = w_j^i$, and extend each f_i to v by setting $f_i(v) = v'$. The set of constraints of \mathcal{P}' is then defined as

$$\{\langle f_1(\mathbf{s}_1), R_1 \rangle, \dots, \langle f_q(\mathbf{s}_q), R_q \rangle\}.$$

Then let the $q \cdot k$ -ary relation R be defined as follows

$$R = \{(\varphi(f_1(w_1)), \dots, \varphi(f_1(w_k)), \dots, \varphi(f_q(w_1)), \dots, \varphi(f_q(w_k)), \varphi(v')) \mid \varphi \text{ is a solution to } \mathcal{P}'\}.$$

The collection $A'_{v_1}, \dots, A'_{v_k}, \dots, A'_{v_1}, \dots, A'_{v_k}$ cannot be extended to a consistent collection for R , since $A'_{v_1}, \dots, A'_{v_k}$ cannot be extended to a consistent collection for \mathcal{P} . However, we shall show that R satisfies the conditions of Lemma 4.5, and thus derive a contradiction.

For any pair of indices $w_{j_1}^{i_1}, w_{j_2}^{i_2}$, we claim that $(A'_{j_1} \times A'_{j_2}) \cap \text{pr}_{\{w_{j_1}^{i_1}, w_{j_2}^{i_2}\}} R \neq \emptyset$. Since \mathcal{P} is 3-minimal any tuple $(a, b) \in (A'_{j_1} \times A'_{j_2}) \cap S_{w_{j_1} w_{j_2}}$ can be extended to a solution $(a, b, c) \in S_{w_{j_1} w_{j_2} v}$. Furthermore, for this solution, we can construct a corresponding solution φ to \mathcal{P}' , such that $\varphi(f_{j_1}(w_{j_1})) = \varphi_W(w_{j_1})$. Indeed, for any constraint $\langle \mathbf{s}_j, R_j \rangle$, this partial solution can be extended to a tuple \mathbf{a} from R_j . Then we assign values to $f_j(w_1), \dots, f_j(w_k)$ accordingly to \mathbf{a} (the variable that are not in the constraint scope $f_j(\mathbf{s}_j)$ can be assigned values arbitrarily).

Now, by Corollary 4.5 we get a contradiction. \square

4.2 Maroti's reduction

Reductions of the second type will be applied to instances, in which all the domains are as-components, but some of them contain semilattice edges. We will call such instances *semilattice free*.

Maroti in [20] suggested a reduction for CSPs that are invariant under a certain binary operation. Let \mathfrak{A} be a class of finite algebras of similar type closed under subalgebras. Suppose that \mathfrak{A} has a term operation f satisfying the following conditions for some $\mathbb{A} \in \mathfrak{A}$:

1. $f(x, f(x, y)) = f(x, y)$ for any $x, y \in \mathbb{A}$;
2. \mathfrak{A} is closed under retracts via unary polynomials $f(a, x), f(x, a)$;
3. for each $a \in \mathbb{A}$ the mapping $x \mapsto f(a, x)$ is not surjective;
4. the set C of $a \in \mathbb{A}$ such that $x \mapsto f(x, a)$ is surjective generates a proper subalgebra of \mathbb{A} .

Then $\text{CSP}(\mathfrak{A})$ is polynomial time reducible to $\text{CSP}(\mathfrak{A} - \{\mathbb{A}\})$.

As is easily seen, the operation \cdot of a class \mathfrak{A} of conservative algebras of closed under subalgebras and any $\mathbb{A} \in \mathfrak{A}$ satisfies conditions (1),(2). If the operation $a \cdot x$ is surjective for some a , then $a \leq x$ for all $x \in \mathbb{A}$. Therefore the only case when condition (3) is not satisfied is when \mathbb{A} has such a minimal element. Finally, condition (4) is satisfied whenever \mathbb{A} is not semilattice free.

We apply Maroti's reduction only in the case when every domain of the instance is either semilattice free, or is an as-component. In this situation this reduction can be slightly modified. More precisely, we will apply it to all semilattice free domains rather than just one. Below we explain the reduction, and the modifications required. The reduction uses 3 types of constructions.

Let $\mathcal{P} = (V, \delta, \mathcal{C})$ be an instance of $\text{CSP}(\mathfrak{A})$ and $p_v: \delta(v) \rightarrow \delta(v)$, $v \in V$. Mappings p_v , $v \in V$, are said to be *consistent* if for any $\langle \mathbf{s}, R \rangle \in \mathcal{C}$, $\mathbf{s} = (v_1, \dots, v_k)$, and any tuple $\mathbf{a} \in R$ the tuple $(p_{v_1}(\mathbf{a}[1]), \dots, p_{v_k}(\mathbf{a}[k]))$ belongs to R . Mappings p_v are called *permutational* if all of them are permutations, they are called *idempotent* if all of them are idempotent. For consistent idempotent mappings p_v by $p(\mathcal{P})$ we denote the *retraction* of \mathcal{P} , that is, \mathcal{P} restricted to the images of p_v . As is easily seen (see [20]), in this case

\mathcal{P} has a solution if and only if $p(\mathcal{P})$ has. Also, if p_v are consistent non-permutational maps, then there are consistent idempotent maps p'_v of \mathcal{P} obtained by iterating p_v .

The next construction uses a binary idempotent operation \cdot satisfying the identity $x \cdot (x \cdot y) = x \cdot y$. Then $t(\mathcal{P})$ denotes the instance $(V', \delta', \mathcal{C}')$ where

- $V' = \{(v, b) \mid v \in V, b \in \delta(v)\}$ is the set of variables;
- the domains are defined by the rule $\delta'(v, b) = b \cdot \delta(v) = \{b \cdot x \mid x \in \delta(v)\}$;
- \mathcal{C}' contains constraints of two types:
first, for each $v \in V$, it contains the constraint $\langle \mathbf{s}_v, R_v \rangle$ where $\mathbf{s}_v = ((v, b_1), \dots, (v, b_k))$ for some enumeration b_1, \dots, b_k of elements of $\delta(v)$, and $R_v = \{(b_1 \cdot c, \dots, b_k \cdot c) \mid c \in \delta(v)\}$;
second, for every $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$, $\mathbf{s} = (v_1, \dots, v_k)$, and $\mathbf{a} \in R$ there is constraint $D_{C, \mathbf{a}} = \langle \mathbf{s}_{C, \mathbf{a}}, R_{C, \mathbf{a}} \rangle$ given by $\mathbf{s}_{C, \mathbf{a}} = ((v_1, \mathbf{a}[1]), \dots, (v_k, \mathbf{a}[k]))$ and $R_{C, \mathbf{a}} = \{\mathbf{a} \cdot \mathbf{x} \mid \mathbf{x} \in R\}$.

The important property of the problem $t(\mathcal{P})$ is that if it has a solution φ then mappings p_v , $v \in V$, given by $p_v(b) = \varphi(v, b)$ are consistent. If $t(\mathcal{P})$ does not have a solution, \mathcal{P} also does not have a solution (see [20])

We describe the last construction used in the reduction for conservative algebras only. Let B_v be the set of all $b \in \delta(v)$ such that ab is a semilattice edge for no $a \in \delta(v)$. For every such b the mapping $x \cdot b$ is injective, while for any other b it is not. Then let $c(\mathcal{P})$ denote the restriction of \mathcal{P} to the sets B_v .

The reduction then goes as follows. First, solve $c(\mathcal{P})$. If it has a solution, it is also a solution of \mathcal{P} , so assume $c(\mathcal{P})$ has no solution. If $t(\mathcal{P})$ has a solution that is not permutational, then \mathcal{P} has consistent non-permutational mappings, p_v , that can be assumed idempotent. In this case \mathcal{P} has a solution if and only if $p(\mathcal{P})$ has, and can be replaced with this smaller problem, as $\text{sum}(p(\mathcal{P})) < \text{sum}(\mathcal{P})$. It remains to consider the case when $p(\mathcal{P})$ has no solution that gives rise to non-permutational mappings.

In this case, as $c(\mathcal{P})$ has no solution, for any solution φ of \mathcal{P} , there is $v \in V$ such that $\varphi(v) = b \notin B_v$. Then for each variable $w \in V$ and every $\delta(w) - B_w$ we create the instance $t(\mathcal{P})$ with an additional unary constraints $\langle (w, b), (b \cdot d) \rangle$, $b \in \delta(w)$. This implies that for any consistent maps p_v that arise from a solution to such instance, $p_w(b) = b \cdot d$, and therefore, they are not permutational. If there is such a non-permutational collection of consistent mappings, we replace \mathcal{P} with $p(\mathcal{P})$; otherwise we conclude that \mathcal{P} has no solution.

4.3 The algorithm and its running time

Consider an instance $\mathcal{P} = (V, \delta, \mathcal{C})$ of $\text{CSP}(\mathfrak{A})$. Recall that it is called semilattice free if none of $\mathcal{G}(\delta(v))$ contains a semilattice edge. Our algorithm works recursively reducing the domains so that eventually we obtain a semilattice free instance.

First, we show how to solve semilattice free instances. Every edge of $\mathcal{G}(\delta(v))$, $v \in V$, in this case is either majority or affine. Therefore for any $v \in V$ and any $a, b \in \delta(v)$ the operation $m(x, y, z) = h(g(x, y, z), g(y, z, x), g(z, x, y))$ is a majority operation if ab is a majority edge, and is an affine operation if ab is an affine edge. Thus m satisfies the conditions of a *generalized majority-minority* operation, and can be solved by the algorithm from [9].

If \mathcal{P} is not semilattice free, but every domain is an as-component, we apply Maroti's reduction, as described in Section 4.2. This reduction repeatedly reduces the problem to a smaller one, $p(\mathcal{P})$, by finding consistent maps p , and either discovers that \mathcal{P} does not have a solution or produces a problem which is semilattice free or has a proper as-component. It also makes recursion calls with instances $t(\mathcal{P})$ and $c(\mathcal{P})$, each of which is either semilattice free or has a domain with a least element and therefore with a proper as-component.

Finally, if \mathcal{P} has a domain with a proper as-component, we apply the as-component exclusion reduction as described in Section 4.1, and either find a solution or reduce some of the domains. This reduction makes recursive calls with instances in which every domain is an as-component.

The correctness of this algorithm follows from the previous sections, [20], and [9]. Therefore, it remains to prove that the algorithm is polynomial time.

Proposition 4.6 *The algorithm is polynomial time in the size of \mathcal{P} .*

Solving semilattice free instances is polynomial time by [9]. We consider the recursion tree generated by the algorithm. It is easy to see that at every node of the tree the amount of work done by the algorithm is bounded by a polynomial, so is the number of recursive calls. Therefore it suffices to show that the depth of recursion is bounded by a constant.

Let $\text{lev}(\mathcal{P})$ for an instance \mathcal{P} of $\text{CSP}(\mathfrak{A})$ denote the maximal size of a semilattice non-free domain of \mathcal{P} . The following lemma is straightforward.

Lemma 4.7 *Let $\mathcal{P} = (V, \delta, \mathcal{C})$ be an instance of $\text{CSP}(\mathfrak{A})$ such that all $\delta(v)$ are as-components (and therefore do not have a least element). Let also p_v , $v \in V$, be consistent maps for \mathcal{P} . Then $p(\mathcal{P}), t(\mathcal{P}), c(\mathcal{P})$ are instances of $\text{CSP}(\mathfrak{A})$, and $\text{lev}(t(\mathcal{P})), \text{lev}(c(\mathcal{P})) < \text{lev}(\mathcal{P})$;*

We use the following observation:

Suppose there is a constant c such that for any problems \mathcal{P}' and \mathcal{P}'' such that \mathcal{P}'' is a successor of \mathcal{P}' in the recursion tree and the length of the path from \mathcal{P}' to \mathcal{P}'' is at least c , then $\text{lev}(\mathcal{P}'') < \text{lev}(\mathcal{P}')$. Then the recursion tree has depth at most $c \cdot k$ where k is the maximal size of a semilattice non-free algebra in \mathfrak{A} .

We show that the algorithm satisfies the condition above for $c = 2$. Let \mathcal{P}' be the problem being solved at some node of the recursion tree. Suppose first that all the domains of \mathcal{P}' are semilattice-free. Then \mathcal{P}' has no successors and there is nothing to prove. Next, suppose that some domain is not an as-component. Then every child of \mathcal{P}' is of the form \mathcal{P}_{I_j} for some strand I_j . Every domain in a problem like this is an as-component. Note, however, that the size of at least some domains may not decrease at this step, if those domains are already as-components. Finally, suppose that all domains of \mathcal{P}' are as-components. Then every child of \mathcal{P}' has the form $c(\mathcal{P}')$, $t(\mathcal{P}')$, or $\mathcal{P}'_{v,d} = t(\mathcal{P}') \cup \{(\langle v, d \rangle, d)\}$. By Lemma 4.7 the maximal size of semilattice non-free domain of each of these problems is strictly less than that of \mathcal{P}' .

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